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Realizing disjoint degree sequences of span at most two: A tractable discrete tomography problem

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ABSTRACT

We consider the problem of coloring a grid using p colors with the requirement that each row and each column has a specific total number of entries of each color.

Ryser (1957) [20], and independently Gale (1957) [10], obtained a necessary and sufficient condition for the existence of such a coloring when two colors are considered. This characterization yields a linear-time algorithm for constructing the coloring when it exists. Later, Gardner et al. (2000) [11], and Chrobak and Dürr (2001) [5], showed that the problem is NP-hard when $p \geqslant 7$ and $p \geqslant 4$, respectively.

The case $p = 3$ was an open problem for several years and has been recently settled by Dürr et al. (2009) [9]: it is NP-hard too. This grid coloring problem is equivalent to finding disjoint realizations of two degree sequences d_1, d_2 in a complete bipartite graph $K_{X,Y}$. These kinds of questions are well studied when one of the degree sequences has span zero or one, where the *span* of a function is the difference between its maximum and its minimum values. In [4], Chen and Shastri (1989) showed a necessary and sufficient condition for the existence of a coloring when $d_1 + d_2$ restricted to X or Y has span at most one. In terms of discrete tomography this latter condition means that for two colors, the sum of the number of occurrences of these colors in each row is k or $k + 1$, for some integer k.

In the present paper we prove an analog to Chen and Shastri's characterization when $d_1 + d_2$ restricted to X and to Y has span at most two. That is, there exist integers k_1 and k_2 such that the sum of the number of occurrences of two of the colors in each row is $k_1 - 1$, k_1 or $k_1 + 1$, and in each column is $k_2 - 1$, k_2 or $k_2 + 1$. Our characterization relies on a new natural condition called the *total saturation condition* which, when not satisfied, gives a non-existence certificate of such a coloring that can be checked in polynomial time.

1. Introduction

Discrete tomography is devoted to the reconstruction of a finite object from its projections. Since its introduction, discrete tomography has shown deep connections with some classical problems in combinatorics (see for instance [14]). One of these problems involves the coloring of a grid using p colors with the requirement that each row and each column has a specific total number of entries of each color. The case $p = 2$ is the well-known problem of reconstructing a matrix of zeros and ones given each row and column sum (see [2] for a survey). This problem was widely studied by Ryser [20] and Gale [10], who gave a necessary and sufficient condition for the existence of a solution. More recently, Gardner et al. [11] studied the

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general case. They proved that this reconstruction problem is NP-hard when considering at least seven colors. Later, Chrobak and Dürr [5] improved this result by showing that it remains NP-hard when p is at least four. The complexity of the case $p = 3$ was an open problem for several years and has been recently settled by Dürr et al. [9]: it is NP-hard too.

There is a natural equivalence between an $|X| \times |Y|$ grid and the complete bipartite graph $K_{X,Y}$, where each cell of the grid corresponds to an edge of the graph (some aspects of this connection are part of the folklore and were formally studied in [7]). Hence, in the previous grid coloring problem each coloring represents a subgraph of $K_{X,Y}$. In addition, we can represent the color restrictions by p functions $d_0, \ldots, d_{p-1} : X \cup Y \to \mathbb{N}$, which assign to each row and column their respective color requirement. Each of these functions d_i represents the prescribed degree sequence of the subgraph corresponding to color i.

Formally, the *degree* of a vertex v of a graph $G = (V, E)$, written $d_G(v)$, is the number of edges incident to v in G. We denote by $d_G : V \to \mathbb{N}$ the function which assigns to every vertex its degree in G. For a subset F of edges, we denote by d_F the degree function of the graph $H = (V, F)$. The function $f : V \to \mathbb{N}$ is *realizable* in G if there exists $F \subseteq E$ such that $d_F = f$. We refer to F as a *realization* of f in G. We say that f is *uniquely realizable in G* if it has only one realization.

Given $d_0, \ldots, d_{p-1} : V \to \mathbb{N}$, a $(d_0, \ldots, d_{p-1})$-*decomposition* of G is a partition $(F_0, \ldots, F_{p-1})$ of E such that F_i is a realization of d_i, for every $i = 0, \ldots, p-1$. Thus the discrete tomography problem can be restated so as to find a $(d_0, \ldots, d_{p-1})$-decomposition of $K_{X,Y}$. In this context, the result by Dürr et al. shows that deciding the existence of a (d_0, d_1, d_2)-decomposition of $K_{X,Y}$ is NP-hard and hence no good characterization can be expected without extra assumption on functions d_i.

From now on, we will mainly focus on (d_0, d_1, d_2)-decomposition of $K_{X,Y}$.

Being a decomposition necessarily means that $d_0 + d_1 + d_2 = d_G$, where $+$ is the usual addition over functions. We then only need to find disjoint realizations F_1, F_2 of d_1, d_2 in G since the edge set $F_0 = E \setminus (F_1 \cup F_2)$ is indeed a realization of $d_0 = d_G - d_1 - d_2$. When d_1 and d_2 have disjoint realizations, they are said to be *disjointly realizable in G*. Our main purpose in this paper is to find some necessary and sufficient conditions for d_1, d_2 to be disjointly realizable in G. First note that we need that both d_1 and d_2 are realizable in G. We also need that $d_1 + d_2 \leqslant d_G$, this condition being called the *degree condition in G*. Another natural necessary condition is that $d_1 + d_2$ is realizable in G.

The conditions cited above are easy to check, and can be deduced from a well-known characterization of realizable functions in bipartite graphs which is due to Ore. We denote by $G = (X, Y, E)$ the bipartite graph with parts X and Y and edge set E. For $S \subseteq X$, $T \subseteq Y$, $E' \subseteq E$, we write $\overline{S} = X - S$, $\overline{T} = Y - T$, $\overline{E'} = E - E'$ and $E'(S, T)$ the set of edges in E' with ends in S and T. In addition, for a function $f : X \cup Y \to \mathbb{N}$ we write $f(S) = \sum_{x \in S} f(x)$ and $f(T) = \sum_{y \in T} f(y)$.

Lemma 1 (*Ore [18]*). *Let $G = (X, Y, E)$ be a bipartite graph and $f : X \cup Y \to \mathbb{N}$. Then, f is realizable in G if and only if $f(X) = f(Y)$ and $f(S) \leqslant f(\overline{T}) + |E(S, T)|$, for each $S \subseteq X$ and $T \subseteq Y$.*

The following result due to Ryser is a straightforward corollary of Lemma 1. It will be one of the central tools of the proof of our main result.

Lemma 2 (*Ryser [20]*). *Let $G = (X, Y, E)$ be a bipartite graph and let $f : X \cup Y \to \mathbb{N}$ be realizable in G. Suppose there exist a realization F_0 of f and two sets $S \subseteq X$, $T \subseteq Y$ such that $F_0(S, T) = E(S, T)$ and $F_0(\overline{S}, \overline{T}) = \emptyset$. Then every realization F of f satisfies $F(S, T) = E(S, T)$ and $F(\overline{S}, \overline{T}) = \emptyset$.*

2. Functions with bounded span

For every fixed integer k, it was conjectured by Rao and Rao [19] that if $d', d_1 : X \to \mathbb{N}$ are realizable functions in K_X satisfying $d'(x) = d_1(x) + k$ for all x in X, then there exists a realization of d' containing a spanning k-regular subgraph. In [16], Kundu solved the conjecture, showing that if d', d_1 are realizable functions in K_X satisfying $d' = d_1 + d_0$, where the span of d_0 is at most one, then d' can be realized by a graph containing a realization of d_0. By the *span* of a function we mean the difference between its maximum and minimum values. Thus, there exists an integer k such that d_0 takes values k or $k + 1$. An algorithmic method for finding these realizations was given by Kleitman and Wang [15] and a very simple proof when $d_0(x) = 1$ for every x, was given by Lovász [17]. Chen [3] noticed that when considering the integer function $d_2 = |X| - 1 - d'$, an even shorter proof could be obtained. Observe that d_2 is realizable in K_X by taking the complement in K_X of a realization of d'. In addition, as d_0 has span at most one, so does $d_1 + d_2 = |X| - 1 - d_0$.

In what follows, we will always use function d to refer to function $d_1 + d_2$. Then Chen's approach of Kundu's result can be stated as follows.

Theorem 3 (*Kundu [16]*). *Let $d_1, d_2 : X \to \mathbb{N}$ be such that the span of d is at most one. Then d_1, d_2 are disjointly realizable in K_X if and only if d_1, d_2 are realizable in K_X and* $\max_{x \in X} d(x) \leqslant |X| - 1$.

Note that the last requirement simply says that the pair d_1, d_2 satisfies the degree condition in K_X. Later, Anstee observed that a similar argument used in the proof of Theorem 3 also works for the complete bipartite graph $K_{X,Y}$ (see Theorem 10.8 in [2]). An alternative and simpler proof of this result was obtained by Chen and Shastri [4]. It is worth pointing out that all these results deal with the problem of finding matrices of zeros and ones with prescribed row and column sum vectors. Notice however that they can be easily translated to the context of degree functions in graphs using the natural equivalence

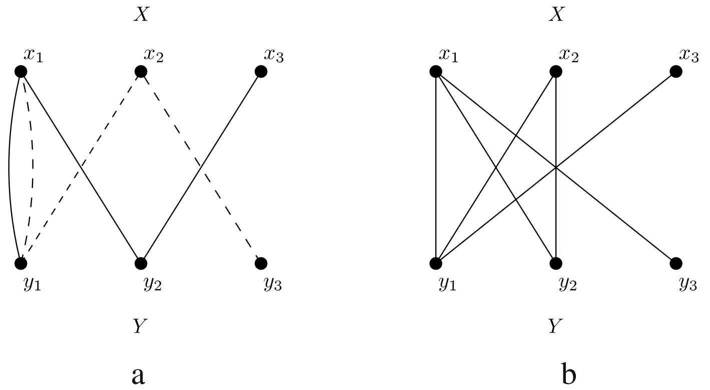

Fig. 1. (a) Realizations of functions d_1 (continuous line) and d_2 (dashed line) in $K_{X,Y}$. (b) Realization of d in $K_{X,Y}$. Because of the degree requirement at each vertex, it can be seen that d_1, d_2 and d are uniquely realizable in $K_{X,Y}$. In particular, x_1y_1 belongs to the unique realization of both d_1 and d_2. Hence d_1, d_2 are not disjointly realizable in $K_{X,Y}$. We remark that both $d|_X$ and $d|_Y$ have span two.

between these matrices (grids) and bipartite graphs that we mentioned in Section 1. The interested reader can find the details in the mentioned references.

Here and subsequently, we use $d|_X$ and $d|_Y$ to refer to the restriction of a function d to subsets X and Y of the vertices of $K_{X,Y}$, respectively, and xy to denote the edge with end vertices $x \in X$ and $y \in Y$. To shorten the notation we write $\max d|_X$ for $\max_{x\in X} d(x)$. Similarly, we use $\max d|_Y$ for $\max_{y\in Y} d(y)$.

Theorem 4 (*Anstee, see Theorem 10.8 in [2]*). *Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ and assume that $d|_X$ has span at most one. Then d_1, d_2 are disjointly realizable in $K_{X,Y}$ if and only if d_1, d_2 are realizable and satisfy the degree condition in $K_{X,Y}$, that is,* $\max d|_X \leqslant |Y|$ *and* $\max d|_Y \leqslant |X|$.

We say that an ordered pair of sets $F = (F_1, F_2)$, with $F_i \subseteq E$ for $i \in \{1, 2\}$, is *a feasible pair for* (d_1, d_2) if for each $i \in \{1, 2\}$, F_i is a realization of d_i. In what follows we use the following notation for a feasible pair $F = (F_1, F_2)$. The union is denoted by $\cup F = F_1 \cup F_2$, the intersection is denoted by $\cap F = F_1 \cap F_2$, the difference is denoted by $F_{1,2} = F_1 \setminus F_2$ or $F_{2,1} = F_2 \setminus F_1$. Finally, the size of the intersection is denoted by $\|F\| = |\cap F|$. For a feasible pair $F = (F_1, F_2)$ and for $i \in \{1, 2\}$, we call a cycle C in a bipartite graph (X, Y, E) an *i-alternating cycle* if its set of edges $E(C)$ alternate between F_i and $E \setminus F_i$.

Let $i \in \{1, 2\}$ and let C be an i-alternating cycle. We define $F^i(C) = (F'_1, F'_2)$ where $F'_j = F_j$ for $j \neq i, j \in \{1, 2\}$ and $F'_i = F_i \setminus E(C) \cup E(C) \setminus F_i$. In other words, we replace the edges of C that are in F_i by those that are in the cycle but not in F_i. Clearly, with previous definitions, F'_i is a realization of d_i and $F^i(C)$ is a feasible pair for (d_1, d_2).

The main idea of Chen and Shastri's proof is based on the following lemma.

Lemma 5 (*Chen and Shastri [4]*). *Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ be realizable functions in $K_{X,Y}$. Assume that for a feasible pair F for d_1, d_2, there exist $x, \bar{x} \in X$ and $y \in Y$ such that $xy \in \cap F$, $\bar{x}y \notin \cup F$ and $d(\bar{x}) > d(x) - 2$. Then there is $y' \in Y$ and $i \in \{1, 2\}$ such that the feasible pair $\|F^i(x, y, \bar{x}, y')\| < \|F\|$. Moreover, F^i is computable in polynomial time.*

Proof. Let $G = (X, Y, \cup F)$ be the bipartite multigraph with parts X and Y where edges in $\cap F$ are counted twice. Hence, $\cap F$ is exactly the set of double edges of G. The total number of edges incident to $\bar{x}$ is at least $d(x) - 1$ and none of them link $\bar{x}$ and y. Since $xy \in \cap F$, the total number of edges incident to x which are not incident to y is $d(x) - 2$. Then there exists a vertex $y' \in Y, y' \neq y$, such that the number of edges between $\bar{x}$ and y' in G is strictly greater than the number of edges between x and y'. Then there is $i \in \{1, 2\}$ such that $\bar{x}y' \in F_i$ and $xy' \notin F_i$. Thus $\{x, y, \bar{x}, y'\}$ is an i-alternating cycle and the feasible pair $F^i(x, y, \bar{x}, y')$ for (d_1, d_2) satisfies $\|F^i(x, y, \bar{x}, y')\| < \|F\|$. □

Note that when d_1, d_2 satisfy the degree condition and the span of $d|_X$ is at most one, for each pair of nondisjoint realizations of d_1, d_2 we can find vertices x, y and $\bar{x}$ satisfying the requirements of Lemma 5. Thus, the proof of the lemma yields a polynomial-time algorithm which starts with two realizations of d_1 and d_2 and reduces the number of double edges. Hence Theorem 4 is a straightforward corollary of this lemma.

In [8], Costa et al. solved a particular case of disjoint realizations of two degree sequences in complete bipartite graphs. Furthermore, Costa et al. studied in [6] the problem when the functions d_0 and d_1 are restricted to have values in $\{0, 2\}$, hence satisfying that $d = d_G - d_0$ has span at most two. Unfortunately, when the function d has span greater than one, the realizability of d_1, d_2 and the degree condition, $d \leqslant d_G$, are not sufficient for d_1, d_2 to be disjointly realizable in $K_{X,Y}$, as shown in the example in Fig. 1. Observe that even asking for d to be realizable in $K_{X,Y}$ is still not a sufficient condition.

Our goal in this work is to provide conditions to establish an equivalent of Theorem 4 when the span of both $d|_X$ and $d|_Y$ is at most two. In the following section we introduce this condition – the total saturation condition – we present our main result, namely Theorem 8, and we show a situation where this condition characterizes the existence of disjoint realizations. We devote Section 4 to the proof of Theorem 8.

3. The total saturation condition

Let $G = (X, Y, E)$ be a bipartite graph and $f : X \cup Y \to \mathbb{N}$ be a realizable function in G. For two sets $S \subseteq X$ and $T \subseteq Y$, we define $m_f(S, T)$ as the minimum number of edges joining S and T over all realizations of function f.

Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ be realizable functions in G. We say that d_1, d_2 *saturate* $E(S, T)$ if $m_{d_1}(S, T) + m_{d_2}(S, T) > |S||T|$. As $|E(S, T)| = |S||T|$ in $K_{X,Y}$, it is easy to see that if d_1, d_2 saturate $E(S, T)$, then d_1, d_2 are not disjointly realizable in G. We say that d_1, d_2 satisfy the *total saturation condition* in G if $m_{d_1}(S, T) + m_{d_2}(S, T) \leqslant |E(S, T)|$, for each $S \subseteq X$ and $T \subseteq Y$. Note that in Fig. 1, the calculation for $S = \{x_1\}$ and $T = \{y_1\}$ gives $m_{d_1}(S, T) + m_{d_2}(S, T) = 2 > |S||T|$, thus d_1, d_2 do not satisfy the total saturation condition in $K_{X,Y}$. As the reader can notice, the total saturation condition says that for every S and T the set $E(S, T)$ is *non-saturated* by the realizations of functions d_1 and d_2. The next theorem shows that for each pair S and T we can check the previous inequality in polynomial time.

Theorem 6. *Let $f : X \cup Y \to \mathbb{N}$ be a realizable function in $G = (X, Y, E)$. For fixed $S \subseteq X$ and $T \subseteq Y$, $m_f(S, T)$ can be calculated in polynomial time.*

Proof. The proof follows from the standard flow formulation given by Gale, which we give in detail (see [10]). We reduce this calculation to a minimum cost flow problem with lower and upper capacities in an auxiliary digraph D. Hence $m_f(S, T)$ is computable in polynomial time (see for instance [1]). We define $D = (V, A)$ as the digraph with vertex set $V = X \cup Y \cup \{s, t\}$ and arcs (s, x) for each $x \in X$, (y, t) for each $y \in Y$ and (x, y) for each $xy \in E$ with $x \in X$ and $y \in Y$.

Let $u, l : A \to \mathbb{N}$ be the lower and upper capacity functions given by $u(s, x) = l(s, x) = f(x)$ for each $x \in X$, $u(y, t) = l(y, t) = f(y)$ for each $y \in Y$, and $u(a) = 1$, $l(a) = 0$ for all the remaining arcs $a \in A$. For $S \subseteq X$ and $T \subseteq Y$, we define a cost function $w : A \to \{0, 1\}$ by $w(x, y) = 1$ if and only if (x, y) is an arc with both $x \in S$ and $y \in T$. The cost of an (s, t)-flow z is defined by $w(z) = \sum_{a \in A} z(a)w(a)$.

Given a realization B of f in G we define $z_B : A \to \mathbb{N}$ by $z_B(s, x) = f(x)$ for every $x \in X$, $z_B(y, t) = f(y)$ for $y \in Y$, and $z_B(x, y)$ with value 1 or 0 depending on whether xy belongs or not to B. Note that $l \leqslant z_B \leqslant u$ and hence z_B is a feasible (s, t)-flow with value $|z_B| = f(X)$. Also note that $w(z_B) = \sum_{a \in A} z_B(a)w(a) = |B(S, T)|$ and thus $w(z_B) \leqslant w(z_{B'})$ if and only if $|B(S, T)| \leqslant |B'(S, T)|$.

Furthermore, since l, u and w are integer-valued functions the integrality theorem for minimum cost flows ensures the existence of an integer minimum cost (s, t)-flow z which is feasible with value $|z| = f(X)$. Define $\tilde{B} = \{xy : (x, y) \in A \text{ with } x \in X, y \in Y \text{ and } z(x, y) > 0\}$. Note that z takes only values 0 or 1 for each $(x, y) \in A$ with $x \neq s$ or $y \neq t$, since for these arcs $0 \leqslant l \leqslant u \leqslant 1$. As the value of z is $f(X)$, $\tilde{B}$ is a realization of f. By our previous observation and since z is a minimum cost (s, t)-flow, we have $m_f(S, T) = |\tilde{B}(S, T)|$. □

We remark that since we have an exponential number of inequalities to check, this result does not entail a polynomial-time algorithm for the satisfiability of the total saturation condition. However, one of the authors has recently proved that it can actually be checked by solving a linear program, and hence it is a problem solvable in polynomial time [13].

Despite the fact that the realizability of d cannot be derived from the realizability of d_1 and d_2, in Theorem 7, we show that the realizability of d easily follows from the realizability of d_1, d_2 when the total saturation condition holds.

Theorem 7. *Let d_1 and d_2 be realizable in $G = (X, Y, E)$. If d_1, d_2 satisfy the total saturation condition in G, then d is realizable in G. In particular d_1 and d_2 satisfy the degree condition.*

Proof. The realizability of d_1, d_2 in G gives, by definition $d(X) = d_1(X)+d_2(X) = d_1(Y)+d_2(Y) = d(Y)$. Let F_i be a realization of d_i in G, where $i = 1, 2$. It is clear that d_i is realizable in $G_i = (X, Y, F_i)$. Thus Lemma 1 shows that $d_i(S) \leqslant d_i(\overline{T}) + |F_i(S, T)|$ for each $S \subseteq X$ and $T \subseteq Y$. Since this holds for every realization of d_i we obtain $d_i(S) \leqslant d_i(\overline{T}) + m_{d_i}(S, T)$. Thus, for each $S \subseteq X, T \subseteq Y$ and $i \in \{1, 2\}$

$$\begin{aligned} d(S) &= d_1(S) + d_2(S) \\ &\leqslant d_1(\overline{T}) + d_2(\overline{T}) + m_{d_1}(S, T) + m_{d_2}(S, T) \\ &\leqslant d(\overline{T}) + |E(S, T)| \end{aligned}$$

where the last inequality follows from the total saturation condition. Again by Lemma 1, d is realizable in G. The last remark comes from the realizability of d in G. □

Consider $d_1, d_2 : X \cup Y \to \mathbb{N}$ such that both $d|_X$ and $d|_Y$ have span at most two. From the above discussion, the realizability and the total saturation condition are necessary for d_1, d_2 to be disjointly realizable in $K_{X,Y} = (X, Y, E)$. Our main result, proved in Section 4, shows that they are also sufficient conditions.

Theorem 8. *Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ such that both $d|_X$ and $d|_Y$ have span at most two. Then, d_1, d_2 are disjointly realizable in $K_{X,Y}$ if and only if d_1, d_2 are realizable and satisfy the total saturation condition in $K_{X,Y}$.*

In the rest of this section we motivate a little bit more the introduction of the total saturation condition. In Theorem 9, we illustrate how the total saturation condition can provide in some cases a necessary and sufficient condition for the existence of a disjoint realization of d_1 and d_2. The proof follows easily from Lemma 1.

Theorem 9. *Let $d_1, d_2 : X \cup Y \to \mathbb{N}$ be realizable in $G = (X, Y, E)$ and assume that d_1 is uniquely realizable. If d_1, d_2 satisfy the total saturation condition in G then they are disjointly realizable.*

Proof. For the sake of contradiction, assume that d_1, d_2 are not disjointly realizable in G and let $F_1 \subseteq E$ be the unique realization of d_1. Clearly, d_2 is not realizable in the graph $H = (X, Y, \overline{F_1})$. Since $d_2(X) = d_2(Y)$, by Lemma 1, there exist $S \subseteq X$ and $T \subseteq Y$ such that

$$d_2(\overline{T}) + |\overline{F_1}(S, T)| < d_2(S). \tag{1}$$

We consider a realization F_2 of d_2 in G such that $m_{d_2}(S, T) = |F_2(S, T)|$. Then,

$$d_2(S) = |F_2(S, T)| + |F_2(S, \overline{T})| \leqslant m_{d_2}(S, T) + d_2(\overline{T}). \tag{2}$$

From Eqs. (1) and (2) we obtain $m_{d_2}(S, T) \geqslant d_2(S) - d_2(\overline{T}) > |\overline{F_1}(S, T)|$. Recall that d_1 is uniquely realizable and hence $m_{d_1}(S, T) = |F_1(S, T)|$. Finally, we obtain $m_{d_1}(S, T) + m_{d_2}(S, T) > |F_1(S, T)| + |\overline{F_1}(S, T)| = |E(S, T)|$, which violates the total saturation condition. □

Note that the example in Fig. 1 shows that only asking for realizability is not enough, even when d_1, d_2 and d are all uniquely realizable.

4. The proof of Theorem 8

Consider d_1 and d_2 both realizable in $K_{X,Y}$ and let F be a feasible pair for d_1, d_2, chosen in such a way that $\|F\|$ is as small as possible. Such a feasible pair is called a *minimal pair*. An edge in $\cap F$ will be called a *double edge* of F.

For a vertex $z \in X \cup Y$, we define $N_F(z) = \{u : zu \in \cup F\}$, $N_F^i(z) = \{u : zu \in F_i\}$, for each $i \in \{1, 2\}$, $M_F^z = Y \setminus N_F(z)$, when $z \in X$, and $M_F^z = X \setminus N_F(z)$, when $z \in Y$. When no confusion arises we use $N(z)$, $N^1(z)$, $N^2(z)$ and M^z instead of $N_F(z)$, $N_F^1(z)$, $N_F^2(z)$ and M_F^z, respectively.

Here is an outline of the proof of Theorem 8. For the sake of contradiction, we shall assume the existence of a minimal pair F such that $\|F\| > 0$. From this assumption, and for an edge $xy \in \cap F$, we will obtain partitions $\{M^y, S, S_1, S_2\}$ of X and $\{M^x, T, T_1, T_2\}$ of Y, such that, for each pair of distinct $i, j \in \{1, 2\}$, edges between S_i and $\overline{T_j}$ (resp. T_i and $\overline{S_j}$) belong to $F_i \setminus F_j$. Finally, by using Lemma 2 we shall deduce that d_1 and d_2 saturate $E(S, T)$.

Before giving the complete proof, we present in Lemmas 10 and 11 some properties and constructions of minimal pairs for d_1 and d_2.

Remember some notations from Section 2: we use d to denote $d_1 + d_2$ and $d|_A$ to denote the restriction of d to the set A. For a feasible pair $F = (F_1, F_2)$ we also use the following notations. $\cup F := F_1 \cup F_2$, $\cap F := F_1 \cap F_2$, $F_{1,2} := F_1 \setminus F_2$, $F_{2,1} := F_2 \setminus F_1$ and $\|F\| := |\cap F|$.

Lemma 10. *Let d_1 and d_2 be realizable with both $d|_X$ and $d|_Y$ having span at most two. Let us assume that d_1, d_2 satisfy the total saturation condition in $K_{X,Y}$ and that F is a minimal pair for d_1 and d_2 such that $\|F\| > 0$. Then, for each $xy \in \cap F$ we have the following properties.*

(1) *$M^x, M^y \neq \emptyset$.*
(2) *$d(x) = \max d|_X$ and $d(y) = \max d|_Y$.*
(3) *For every $\bar{x} \in M^y$, $\bar{y} \in M^x$, we have that $d(\bar{x}) = d(x) - 2$ and $d(\bar{y}) = d(y) - 2$.*
(4) *For every $\bar{x} \in M^y$, $\bar{y} \in M^x$, we have that $M^x \cup \{y\} = M^{\bar{x}}$ and $M^y \cup \{x\} = M^{\bar{y}}$.*
(5) *Vertex x (resp. y) is incident with exactly one edge in $\cap F$.*

Proof. As we have seen in Section 3, if d_1, d_2 do not satisfy the degree condition then the total saturation condition does not hold. Hence, $\max d|_X \leqslant |Y|$ and $\max d|_Y \leqslant |X|$.

(1) Since $xy \in \cap F$ we have that $|N(x)| < d(x) \leqslant |Y|$ and $|N(y)| < d(y) \leqslant |X|$. Then, $M^x, M^y \neq \emptyset$.
(2) Since $d|_X$ has span at most two, we have $\max d|_X \leqslant \min d|_X + 2$. By Lemma 5, for each $\bar{x} \in M^y$ it follows that $d(x) \geqslant d(\bar{x}) + 2$; otherwise there exists a feasible pair F' with $\|F'\| < \|F\|$, which contradicts that F is a minimal pair. Thus $\max d|_X \geqslant d(x) \geqslant d(\bar{x}) + 2 \geqslant \min d|_X + 2$. Therefore, $\max d|_X = d(x)$. A similar argument shows that $\max d|_Y = d(y)$.
(3) By previous equalities we also have that $d(x) = d(\bar{x}) + 2$, for every $\bar{x} \in M^y$. Again, by a similar argument, we get that $d(y) = d(\bar{y}) + 2$, for every $\bar{y} \in M^x$.
(4) Let $\bar{x} \in M^y$. We first prove that $M^x \cup \{y\} \subseteq M^{\bar{x}}$. Clearly, $y \in M^{\bar{x}}$. Let $\bar{y} \in M^x$. Then $\bar{x}\bar{y} \notin F_1$, otherwise $F' := F^1(x, y, \bar{x}, \bar{y})$ is a feasible pair for (d_1, d_2) and $\|F'\| < \|F\|$, contradicting the minimality of F. A symmetric argument shows that $\bar{x}\bar{y} \notin F_2$. Hence, $\bar{y} \in M^{\bar{x}}$.

   By (2) and (3), $\bar{x}$ is not incident to any edge in $\cap F$. Then $|N(\bar{x})| = d(\bar{x})$. Hence, $|M^{\bar{x}}| = |Y| - d(\bar{x}) = |Y| - (d(x) - 2) \leqslant |Y| + 1 - |N(x)| = |M^x| + 1$. However, we already showed that $M^x \cup \{y\} \subseteq M^{\bar{x}}$. Then by the previous inequality and since $y \notin M^x$, we conclude that $M^x \cup \{y\} = M^{\bar{x}}$. A similar argument shows that $M^y \cup \{x\} = M^{\bar{y}}$.

(5) From (4) we have that $N(x) = N(\bar{x}) \cup \{y\}$. Then using the fact that $\bar{x} \in M^y$ is not incident to any double edge and (3), $|N(x)| = |N(\bar{x})| + 1 = d(\bar{x}) + 1 = d(x) - 1$. This shows that x is incident to exactly one edge in $\cap F$. A similar argument works for y. $\square$

Let F be a minimal pair for d_1 and d_2 and $e = xy \in \cap F$. We say that a pair (F', e') *is a child of* (F, e), if there are distinct $i, j \in \{1, 2\}$, such that $e' \in F_j \setminus F_i$, and e' satisfies one of the following conditions:

- $e' = xy'$, $y' \neq y$ and $F' = F^i(x, y, \bar{x}, y')$, for some $\bar{x} \in M^y \cap N^i(y')$;
- $e' = x'y$, $x' \neq x$ and $F' = F^i(x, y, x', \bar{y})$, for some $\bar{y} \in M^x \cap N^i(x')$;
- $e' = x'y'$, $x' \neq x$, $y' \neq y$ and $F' = F^i(x, y, \bar{x}, y', x', \bar{y})$, for some $\bar{y} \in M^x \cap N^i(x')$ and some $\bar{x} \in M^y \cap N^i(y')$.

We say that (F', e') *is a descendant of* (F, e), denoted by $(F, e) \to (F', e')$, if either $(F', e') = (F, e)$ or there exist edges $e_0, \ldots, e_t$ and $F^0, \ldots, F^t$ feasible pairs satisfying $(F^0, e_0) = (F, e)$, $(F^t, e_t) = (F', e')$ and such that (F^k, e_k) is a child of (F^{k-1}, e_{k-1}), for each $k \in \{1, \ldots, t\}$. If $e' = x'y'$ we say that x' and y' are descendants of e. We denote by $D(e)$ the set of all descendants of e.

Lemma 11. *Let d_1 and d_2 be realizable with both $d|_X$ and $d|_Y$ having span at most two and d_1, d_2 satisfying the total saturation condition in $K_{X,Y}$. Let F be a minimal pair for d_1 and d_2 and for $xy \in \cap F$, let $(F', x'y')$ be a descendant of (F, xy). Then*

(1) *F' is a minimal pair and $x'y' \in \cap F'$.*
(2) *$d(x') = \max d|_X$ and $d(y') = \max d|_Y$.*
(3) *$M^{y'}_{F'} = M^y_F$ and $M^{x'}_{F'} = M^x_F$.*
(4) *If $x' \neq x$ (resp. $y' \neq y$), then the set M^x (resp. M^y) is a singleton.*
(5) *If $z \notin D(xy) \cup M^x_F \cup M^y_F$, then for each $i \in \{1, 2\}$, $N^i_{F'}(z) = N^i_F(z)$.*

Proof. Let $e_0, \ldots, e_t$ and $F^0, \ldots, F^t$ satisfy $(F^0, e_0) = (F, xy)$, $(F^t, e_t) = (F', x'y')$ such that (F^k, e_k) is a child of (F^{k-1}, e_{k-1}), for each $k \in \{1, \ldots, t\}$. All the properties will be proved by induction on t. We first consider the case $t = 1$. We only consider the first situation, that is, when $e' = xy'$ for $y' \neq y$. The remaining cases can be proved similarly.

(1) Observe that the set $\{x, y, \bar{x}, y'\}$ is an i-alternating cycle of length 4. Hence, F' is a feasible pair for d_1 and d_2. In addition, $xy' \in \cap F'$, $xy \in F'_{1,2} \cup F'_{2,1}$ and $\bar{x}y'$ does not belong to $\cup F'$. Thus, $\|F'\| = \|F\|$. Consequently F' is also a minimal pair.
(2) From (1), $x'y'$ is a double edge of the minimal pair F'. Then by (2) in Lemma 10 applied to x' and y' in F' we have that $d(x') = \max d|_X$ and $d(y') = \max d|_Y$.
(3) It is clear that $M^x_{F'} = M^x_F$. Let $\bar{y} \in M^x_F$. It is clear from the construction that $M^{\bar{y}}_{F'} = M^{\bar{y}}_F$. Then, by (4) in Lemma 10 applied to y' in F' and to y in F, we have that $M^{y'}_{F'} \cup \{x\} = M^{\bar{y}}_{F'} = M^{\bar{y}}_F = M^y_F \cup \{x\}$.
(4) If there is $z \in M^y_F$ with $z \neq \bar{x}$, then by (4) in Lemma 10 $zy' \in \cup F$. Also the edges incident to z in F and in F' are the same. Hence, $z \notin M^{y'}_{F'}$, contradicting the previous property.
(5) It is clear that for each $z \notin \{x, x', y, y'\} \cup M^x \cup M^y$, and each $i \in \{1, 2\}$, $N^i_{F'}(z) = N^i_F(z)$.

It is clear that for $t \geqslant 2$, the properties follow from an induction argument: the step from $t - 1$ to t corresponds to the situation where (F^t, e_t) is a child of (F^{t-1}, e_{t-1}). $\square$

Proof of Theorem 8. The forward implication is easy. In order to prove the backward implication, let us assume, for the sake of contradiction, that d_1 and d_2 are both realizable in $K_{X,Y}$ but not disjointly realizable.

For the rest of the proof we fix a minimal pair F and $xy \in \cap F$, with $x \in X$ and $y \in Y$.

Let $S = D(xy) \cap X$, $T = D(xy) \cap Y$, and for each $i \in \{1, 2\}$, let $S_i = N^i(y) \setminus S$ and $T_i = N^i(x) \setminus T$.

From (2) in Lemma 11, we know that each vertex in S has degree $\max d|_X$, and from (3) in Lemma 10, that no vertex in M^y has degree $\max d|_X$. Then, $S \subseteq N(y)$. Similarly, we can prove that $T \subseteq N(x)$. Therefore, $\{M^y, S, S_1, S_2\}$ is a partition of X and $\{M^x, T, T_1, T_2\}$ is a partition of Y.

We prove that $E(S_i, \overline{T_j}) \cup E(\overline{S_j}, T_i) \subseteq F_{i,j}$, for each pair of distinct $i, j \in \{1, 2\}$. To ease the presentation we only prove that $E(S_1, M^x \cup T \cup T_1) \subseteq F_{1,2}$. The other cases can be obtained by a similar argument.

We first prove that $E(S_1, M^x) \subseteq F_{1,2}$. Let $z \in S_1$ and $\bar{y} \in M^x$. By the definition of S_1 we get that $S_1 \subseteq N(y)$ and from (4) in Lemma 10 applied to $\bar{y}$ we know that $M^{\bar{y}} = M^y \cup \{x\}$, or equivalently, $N(\bar{y}) \cup \{x\} = N(y)$. Since $x \notin S_1$ we conclude that $z \in N(\bar{y})$, and then $z\bar{y} \in \cup F$. From (5) in Lemma 10, vertex y is incident with exactly one double edge in $\cap F$. However, $z \neq x$ and $z \in S_1 \subseteq N^1(y)$, which implies that $yz \notin F_2$. If $z\bar{y} \in F_2$, then $C = \{x, y, z, \bar{y}\}$ is a 2-alternating cycle for which $(F, xy) \to (F^2(C), zy)$ getting the contradiction: $z \in S$. Then, $z\bar{y} \in F_{1,2}$.

By a symmetric argument, it follows that $E(T_1, M^y) \subseteq F_{1,2}$.

We now prove that $E(S_1, T_1) \subseteq F_{1,2}$. Let $z \in S_1$, $z' \in T_1$, $\bar{y} \in M^x$ and $\bar{x} \in M^y$. From previous analysis, we know that $z\bar{y}, \bar{x}z' \in F_1$. If $zz' \notin \cup F$ then $C = \{x, y, \bar{x}, z', z, \bar{y}\}$ is a 1-alternating cycle with $\|F^1(C)\| < \|F\|$ which contradicts the minimality of F. Hence, $zz' \in \cup F$. If $zz' \in F_{2,1}$, then $(F, xy) \to (F^1(C), zz')$ contrary to the fact that $z \in S_1$ and $z' \in T_1$. Hence, $zz' \in F_1$. If zz' is double in F then (1) in Lemma 10 shows that there is $w \in X$, $w \neq x$, such that $wz' \notin \cup F$. By (4) in Lemma 10 applied to zz' and $w \in M^{z'}$, $w\bar{y} \in \cup F$. As $\bar{x}z' \in F_1$, we know that $w\bar{y} \in F_2$. Otherwise, $C' = \{x, y, \bar{x}, z', w, \bar{y}\}$ is a 1-alternating cycle for which $\|F^1(C')\| < \|F\|$. But then, $C'' = \{x, y, z, z', w, \bar{y}\}$ is a 2-alternating cycle for which $\|F^2(C'')\| < \|F\|$: in $F^2(C'')$, xy and zz' are no longer double edges and we only add zy as double edge. Therefore, $zz' \in F_{1,2}$.

We now prove that $E(S_1, T) \subseteq F_{1,2}$. Let $z \in S_1$ and $z' \in T$. Let $x' \in S$ and F' be a minimal pair such that $(F, xy) \to (F', x'z')$. From (5) in Lemma 11, as z is not a descendant of x, we know that $N^i_{F'}(z) = N^i_F(z)$, for each $i \in \{1, 2\}$. So we only need to show that $zz' \in F'_{1,2}$. First, from (5) in Lemma 10 applied to the double edge $x'z'$ in $\cup F'$, zz' is not a double edge in F'. Let us consider $\overline{y} \in M^x_F$. Since $z \in S_1$, a previous case shows that $z\overline{y} \in F_{1,2}$, and hence $z\overline{y} \in F'_{1,2}$. Moreover, from (3) in Lemma 11 we have that $M^{x'}_{F'} = M^x_F$; then $x'\overline{y} \notin \cup F'$. If $zz' \notin F'_1$, then $C = \{x', z', z, \overline{y}\}$ is a 1-alternating cycle in F'. In case $zz' \in F'_2$ we obtain that zz' is double in $F'^1(C)$; a contradiction with the fact that z is not a descendant of xy. Otherwise we obtain that $\|F'^1(C)\| < \|F'\|$, which contradicts the fact that F' is a minimal pair.

To end the proof we now show that d_1 and d_2 saturate $E(S, T)$.

Let F' be any *feasible pair* for d_1 and d_2. For each pair of distinct $i, j \in \{1, 2\}$, Lemma 2, applied to F_i, implies that $E(S_i, \overline{T_j}) \subseteq F'_i$ and $E(\overline{S_i}, T_j) \cap F'_i = \emptyset$ and that $E(\overline{S_i}, T_j) \subseteq F'_j$ and $E(S_i, \overline{T_j}) \cap F'_j = \emptyset$.

Clearly $|F'_i(S, M^x)| \leqslant d_i(M^x) - |F'_i(S_1 \cup S_2, M^x)|$, for each $i \in \{1, 2\}$. Since $M^x \subseteq \overline{T}_1$ and $M^x \subseteq \overline{T}_2$, we get, for each $i \in \{1, 2\}$, $|F'_i(S_1 \cup S_2, M^x)| = |F'_i(S_i, M^x)| = |S_i||M^x|$ and hence

$$\begin{aligned} |F'_1(S, M^x)| + |F'_2(S, M^x)| &\leqslant d(M^x) - |S_1||M^x| - |S_2||M^x| \\ &= (|X| - |M^y| - 1)|M^x| - (|S_1| + |S_2|)|M^x| \\ &= (|S| - 1)|M^x| \\ &< |S| \end{aligned}$$

where the first equality follows since, from (3) in Lemma 10, the vertices in M^x have degree $\min d|_Y = |X| - |M^y| - 1$; the second one comes from the fact that $X = M^y \cup S \cup S_1 \cup S_2$; the last inequality comes from (4) in Lemma 11: either S or M^x is a singleton.

We also have the following inequalities.

$$\begin{aligned} |S||T| &= |S|(|Y| - |M^x| - |T_1| - |T_2|) \\ &= |S|(|Y| + 1 - |M^x|) - |S|(|T_1| + |T_2| + 1) \\ &= d(S) - |S|(|T_1| + |T_2| + 1) \\ &= d_1(S) - |S||T_1| + d_2(S) - |S||T_2| - |S| \\ &= F'_1(S, T) + F'_2(S, T) + F'_1(S, M^x) + F'_2(S, M^x) - |S| \\ &< F'_1(S, T) + F'_2(S, T) \end{aligned}$$

where the first equality comes from the fact that $Y = M^x \cup T \cup T_1 \cup T_2$; the second equality is easy; the third comes from (2) in Lemma 11: vertices in $D(xy)$ have maximum degree in its respective parts; the fourth one is easy; the previous to the last equality follows from the fact that $S \cap S_1$ and $S \cap S_2$ are empty sets and then, for each $i \in \{1, 2\}$, $|F'_i(S, T_1 \cup T_2)| = |S||T_i|$ which implies that $d_i(S) - |S||T_i| = |F'_i(S, T)| + |F'_i(S, M^x)|$; the last inequality was already proved.

Since this holds for each feasible pair F' for d_1 and d_2 we conclude that $m_{d_1}(S, T) + m_{d_2}(S, T) > |S||T|$. □

It would be tempting to propose the realizability and the satisfiability of the total saturation condition as necessary and sufficient conditions for the general case of two functions in $K_{X,Y}$. Unfortunately, this is not the case: using the NP-hardness proof by Dürr et al. [9] it is possible to construct a pair of functions d_1, d_2 which satisfy these necessary conditions in $K_{X,Y}$ but that are not disjointly realizable [12].

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